

B -meson distribution amplitudes*

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Abstract

B -meson light-cone distribution amplitudes are discussed in these lectures in the framework of HQET. The evolution equation for the leading-twist distribution amplitude is derived in one-loop approximation. QCD sum rules for distribution amplitudes are discussed.

1 Introduction

Many exclusive B -decay amplitudes in the framework of SCET contain B -meson distribution amplitudes [1]. The amplitude of the decay $B \rightarrow \gamma l \bar{\nu}$ at large photon energies is given, up to power corrections, by a convolution of a hard part (perturbatively calculable) and the B -meson distribution amplitude [2]. Amplitudes of some decays, e.g., $B \rightarrow \pi l \bar{\nu}$ at large pion energies, contain both factorizable and non-factorizable contributions [3, 4]. Factorizable parts of decay amplitudes contain light-cone distribution amplitudes of the initial B -meson and final hadron(s). They describe large-distance (soft) structure of these hadrons, and cannot be calculated in perturbation theory. The theory of hadronic distribution amplitudes in QCD is reviewed in [5].

Quark–antiquark distribution amplitudes of B -meson in HQET were introduced and investigated in [6]. They are defined as Fourier transforms of matrix elements of some gauge-invariant bilocal operators between B -meson and vacuum. Renormalization of these operators was calculated in one-loop approximation. However, an unusual term $1/\varepsilon^2$ in the one-loop renormalization constants was erroneously omitted in [6]. The correct evolution equation for the leading-twist distribution amplitude was derived in [7] at one loop. The evolution kernel contains, in addition to terms obtained earlier [6], an unusual term $\log(\omega/\mu)\delta(\omega - \omega')$. The method of solution of the evolution equation is also discussed in [7] in detail.

Quark–antiquark–gluon distribution amplitudes of B -meson and their relations to quark–antiquark ones (based on equations of motion) are discussed in [8].

Sum rules for the quark–antiquark distribution amplitudes were obtained in [6]. A simple model of these distribution amplitudes at a low normalization scale (of order of

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hadronic scale) was proposed. Radiative corrections to the perturbative term and the quark-condensate term were later calculated [9].

In these lectures, we first briefly discuss what is HQET¹ (Sect. 2). A much more detailed presentation can be found in the textbooks [10, 11]. After a short discussion of f_B (Sect 3), quark–antiquark distribution amplitudes are introduced (Sect. 4). Quark–antiquark–gluon distribution amplitudes and their relations to two-particle ones are discussed in Sect. 5. Sect. 6 is the central (and longest) one. Here renormalization of light-cone bilocal quark operators in HQET is considered in one-loop approximation. A detailed derivation of the evolution kernel is presented, based on the methods of [12]. Finally, sum rules for the distribution amplitudes are briefly discussed in Sect. 7.

2 Heavy Quark Effective Theory

Let's consider a heavy antiquark with mass m and momentum

$$p = mv + k \quad (2.1)$$

in QCD. Here v is a fixed 4-velocity ($v^2 = 1$), and the residual momentum k , as well as momenta of all light quarks and gluons, are supposed to be small (compared to m). The propagator of this heavy antiquark can be written as

$$\frac{m - m\not{v} - \not{k}}{(mv + k)^2 - m^2 + i0} = \frac{1 - \not{v}}{2} \frac{1}{k \cdot v + i0} + \mathcal{O}\left(\frac{1}{m}\right). \quad (2.2)$$

The leading term here is the HQET propagator. This can be graphically presented as²

$$\text{---}\overleftarrow{\text{---}}\text{---} = \text{---}\overleftarrow{\text{---}}\text{---} + \mathcal{O}\left(\frac{1}{m}\right). \quad (2.3)$$

$mv + k$ $mv + k$

In a vertex sandwiched between such propagators, we may substitute

$$\frac{1 - \not{v}}{2} \gamma^\mu \frac{1 - \not{v}}{2} = \frac{1 - \not{v}}{2} (-v^\mu) \frac{1 - \not{v}}{2}. \quad (2.4)$$

Projector can also be inserted near external legs, so that all QCD vertices can be replaced by the HQET ones,

$$\text{---}\overleftarrow{\text{---}}\text{---} \begin{matrix} \mu \\ \text{---} \\ a \end{matrix} = ig_0 t^a (-v^\mu), \quad (2.5)$$

up to $\mathcal{O}(k/m)$ corrections.

¹Unlike most texts on HQET, we consider not a heavy quark but a heavy antiquark. Of course, this makes no difference, but the active participant in the distribution amplitudes is the light quark (and, possibly, a gluon etc.), and this choice makes notation more natural.

²The arrow here is somewhat misleading: there is a *particle* called antiquark and propagating from left to right; it has no antiparticle.

These Feynman rules can be obtained from the Lagrangian

$$L = \bar{Q}_v i v \cdot \overleftarrow{D} Q_v + (\text{light fields}) \quad (2.6)$$

The antiquark field is our main field here, it satisfies

$$\bar{Q}_v \not{v} = -\bar{Q}_v; \quad (2.7)$$

Q_v is the conjugate field for \bar{Q}_v . Here the covariant derivatives are

$$D_\mu q = (\partial_\mu - i A_\mu) q, \quad \bar{q} \overleftarrow{D}_\mu = \bar{q} \left(\overleftarrow{\partial}_\mu + i A_\mu \right), \quad A_\mu = g_0 A_{0\mu}^a t^a. \quad (2.8)$$

QCD tree diagrams are reproduced by HQET up to $\mathcal{O}(k_i/m)$ corrections (they can also be reproduced, if we add the appropriate $1/m$ terms into the Lagrangian).

The heavy quark chromomagnetic moment is $\sim 1/m$ by dimensionality. At the leading order in $1/m$, the heavy-quark spin does not interact with the gluon field. Therefore, it may be rotated at will, without changing the physics (heavy-quark spin symmetry). It may even be switched off (superflavour symmetry). We shall work with the spinless heavy antiquark,

$$L = Q_v^* i v \cdot \overleftarrow{D} Q_v + (\text{light fields}), \quad (2.9)$$

during most of these lectures, because this greatly simplifies reasoning and calculations. Here again Q_v^* is the main (scalar) field, and Q_v is its conjugate.

So, tree QCD diagrams, expanded in k_i/m to some order, are reproduced by the corresponding HQET diagrams. But what about loops? Here things are not so simple [13]. Let's consider, for example, the heavy-light two-point diagram (Fig. 1) with $p = mv + k$, where the residual momentum k is small. By choosing v along p we can always ensure $k = \omega v$. Let's consider the integral

$$I = \frac{-im^2}{\pi^{d/2}} \int \frac{d^d l}{[m^2 - (mv + k + l)^2](-l^2)}, \quad (2.10)$$

which has neither ultraviolet (UV) nor infrared (IR) divergences. There are two regions of the loop momentum l in this integral:

- Hard region $l \sim m$. Expanding the integrand in $k \ll m, l$, we have

$$\frac{1}{[m^2 - (mv + l)^2](-l^2)} + 4 \frac{(m + l \cdot v)\omega}{[m^2 - (mv + l)^2]^3(-l^2)} + \dots \quad (2.11)$$

- Soft region $l \sim \omega$. Expanding the integrand in $k, l \ll m$, we have

$$\frac{1}{[-2m(k + l) \cdot v]^2(-l^2)} + 2 \frac{(k + l)^2}{[-2m(k + l) \cdot v]^3(-l^2)} + \dots \quad (2.12)$$

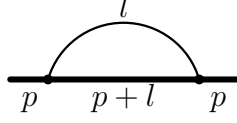


Figure 1: Heavy-light two-point diagram

The contribution of the hard region is

$$I_h = m^{-2\varepsilon} \left[M(2, 1) + 2\frac{\omega}{m} [M(3, 0) - M(2, 1) + 2M(3, 1)] + \dots \right], \quad (2.13)$$

$$\frac{\mu^{2\varepsilon} I_h}{\Gamma(1 + \varepsilon)} = -\frac{1}{2\varepsilon} + \log \frac{m}{\mu} + \left(\frac{1}{\varepsilon} - 2 \log \frac{m}{\mu} - 1 \right) \frac{\omega}{m} + \dots$$

where the on-shell massive two-point integrals (Fig. 2)

$$\int \frac{d^d l}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2} m^{d-2(n_1+n_2)} M(n_1, n_2), \quad (2.14)$$

$$D_1 = m^2 - (l + mv)^2 - i0, \quad D_2 = -l^2 - i0$$

are

$$M(n_1, n_2) = \frac{\Gamma(d - n_1 - 2n_2)\Gamma(-d/2 + n_1 + n_2)}{\Gamma(n_1)\Gamma(d - n_1 - n_2)}. \quad (2.15)$$

This contribution is IR divergent.

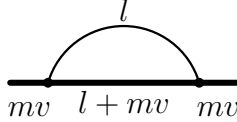


Figure 2: On-shell massive two-point integrals

The contribution of the soft region is

$$I_s = (-2\omega)^{-2\varepsilon} \left[I(2, 1) + \frac{\omega}{m} [I(3, 1) - 2I(2, 1)] + \dots \right], \quad (2.16)$$

$$\frac{\mu^{2\varepsilon} I_s}{\Gamma(1 + \varepsilon)} = \frac{1}{2\varepsilon} - \log \frac{-2\omega}{\mu} - \left(\frac{1}{\varepsilon} - 2 \log \frac{-2\omega}{\mu} - \frac{1}{2} \right) \frac{\omega}{m} + \dots$$

where the HQET two-point integrals (Fig. 3, $\omega = k \cdot v$)

$$\int \frac{d^d l}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2} (-2\omega)^{d-n_1-2n_2} I(n_1, n_2), \quad (2.17)$$

$$D_1 = -2(l + p) \cdot v - i0, \quad D_2 = -l^2 - i0$$

are

$$I(n_1, n_2) = \frac{\Gamma(-d + n_1 + 2n_2)\Gamma(d/2 - n_2)}{\Gamma(n_1)\Gamma(n_2)}. \quad (2.18)$$

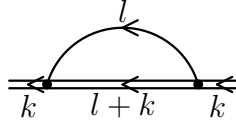


Figure 3: HQET two-point integrals

This contribution is UV divergent.

The complete result is finite:

$$I = -\log \frac{-2\omega}{m} + \left(2 \log \frac{-2\omega}{m} - \frac{1}{2} \right) \frac{\omega}{m} + \dots \quad (2.19)$$

This expansion can be easily continued if desired.

What about higher loops? Let's consider, for example, the two-loop two-point diagram with a small external residual momentum. There are several regions of the loop momenta in this diagram (Fig. 4). In each of them, some momenta are hard ($\sim m$), some are soft ($\sim \omega$). As we have already seen, a heavy-quark line with a soft residual momentum becomes an HQET line. Soft massless lines are shown by dashed lines in the figure.

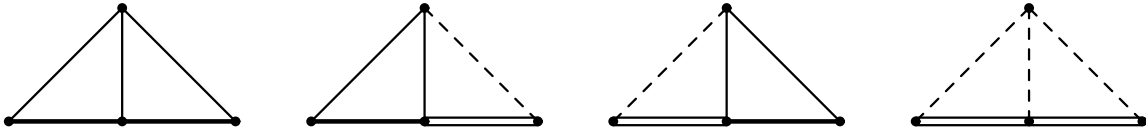


Figure 4: Regions in the two-loop diagram

For example, in the second diagram in Fig. 4, the left loop is hard. It contains a single scale m . All external momenta of this loop are soft, including those of the lines belonging to the other loop. We can expand the integrand in these small momenta; after taking the loop integral, we obtain a polynomial in these momenta. Now from the point of view of the soft loop (large distances) this hard loop is just a local vertex. This loop contains a single scale ω , and we can calculate it, obtaining a non-analytical function of the external residual energy ω .

In a general multiloop diagram, hard lines must always form loops, so that momentum conservation can hold after neglecting all soft momenta. There can be several disconnected hard parts; each of them must contain at least one heavy line (a subdiagram consisting of only light lines and having soft external momenta has no reason to be hard). From the point of view of the soft part, hard parts are just local vertices. There may appear a soft subdiagram connected to the rest of the diagram only at such a vertex. Such a subdiagram is scaleless and hence vanishes. For example, we could consider one more region in Fig. 4: when the only soft line is the middle light line, all the rest are hard. But this soft line forms a loop containing one local vertex (the integrated hard loop), and hence it vanishes.

In the usual HQET formalism, Lagrangian contains local operators multiplied by matching coefficients. QCD operators are also expanded in HQET operators with matching coefficients. These matching coefficients are the only quantities in the theory which depend on the hard scale m . Diagrammatically, they come from hard loops in QCD diagrams. Local operators produce vertices polynomial in their external momenta. They appear in HQET diagrams, which contain only the soft scale ω . These HQET diagrams are soft parts of QCD diagrams.

3 B -meson decay constant

During these lectures, we shall mostly live in a world with a heavy antiquark having $j^P = 0^+$. Physics in the real world is the same, up to $1/m$ corrections. We shall work in the v rest frame.

The ground-state S -wave $\bar{Q}q$ meson has the quantum numbers $j^P = \frac{1}{2}^+$. There are 2 P -wave excited mesons with $j^P = \frac{1}{2}^-$ and $\frac{3}{2}^-$. The heavy-light quark current

$$j = Q_v^* q \quad (3.1)$$

has no definite parity; the currents with parity $P = \pm 1$ are

$$j_P = \frac{1 + P\gamma^0}{2} j. \quad (3.2)$$

They have the quantum numbers of S -wave $\frac{1}{2}^+$ mesons and P -wave $\frac{1}{2}^-$ mesons. The ground-state meson M has a Dirac wave function u which satisfies $\gamma^0 u = u$ and is normalized by $\bar{u}u = 1$. The matrix element of j from M to vacuum is

$$\langle 0 | j | M \rangle = F u, \quad (3.3)$$

where the one-meson states are normalized by the non-relativistic condition

$$\langle M, \vec{p}' | M, \vec{p} \rangle = (2\pi)^3 \delta(\vec{p}' - \vec{p}). \quad (3.4)$$

The correlator of the heavy-light currents (Fig. 5),

$$i \langle T j(x) \bar{j}(0) \rangle = \delta(\vec{x}) \Pi(x^0), \quad (3.5)$$

has the structure

$$\Pi(x^0) = A + B \not{v}. \quad (3.6)$$

Therefore, the correlators of the currents with definite parity are

$$i \langle T j_P(x) \bar{j}_P(0) \rangle = \delta(\vec{x}) \Pi_P(x^0), \quad \Pi_P = A + P B = \frac{1}{4} \text{Tr}(1 + P\gamma^0) \Pi. \quad (3.7)$$

The spectral density of, say, the correlator with $P = +1$ is

$$\rho_+(\varepsilon) = F^2 \delta(\varepsilon - \bar{\Lambda}) + \dots \quad (3.8)$$

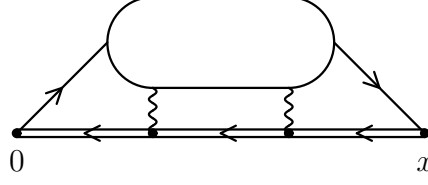


Figure 5: Correlator of heavy-light currents

where $\bar{\Lambda}$ is the residual energy of the ground-state meson, and the dots mean contribution of excited states.

Now we shall return to the real world with $j^P = \frac{1}{2}^-$ heavy antiquark for a while, but still with $m = \infty$. The S -wave ground-state meson turns into a degenerate doublet with $j^P = 0^-, 1^-$. The two P -wave mesons turn into two degenerate doublets: with $j^P = 0^+, 1^+$ and with $j^P = 1^+, 2^+$. All heavy-light currents $\bar{Q}_v \Gamma q$ reduce, due to (2.7), to 4 ones with

$$\Gamma = \gamma_5, \quad \vec{\gamma} \quad \text{and} \quad \Gamma = 1, \quad \vec{\gamma} \gamma_5. \quad (3.9)$$

The first 2 currents, with Γ anticommuting with γ^0 , have the quantum numbers of the ground-state $0^-, 1^-$ mesons; the second 2 currents, with Γ commuting with γ^0 , have the quantum numbers of the P -wave $0^+, 1^+$ mesons.

The correlators (Fig. 5) are

$$i \langle T j_2(x) j_1^+(0) \rangle = \delta(\vec{x}) \Pi_{12}(x^0), \quad \Pi_{12} = \text{Tr} \bar{\Gamma}_1 \frac{1 - \gamma^0}{2} \Gamma_2 \Pi, \quad (3.10)$$

or

$$\Pi_{12} = \Pi_P \text{Tr} \bar{\Gamma}_1 \frac{1 - \gamma^0}{2} \Gamma_2, \quad (3.11)$$

where $P = +1$ for Γ anticommuting with γ^0 and -1 for commuting. For $\Gamma = \gamma_5$ and γ^i , the correlators are $2\Pi_+$ and $2\Pi_+ \delta^{ij}$. Their spectral densities are $F_B^2 \delta(\varepsilon - \bar{\Lambda}) + \dots$ and $F_{B^*}^2 \delta(\varepsilon - \bar{\Lambda}) \delta^{ij} + \dots$, where

$$\langle 0 | \bar{Q}_v \gamma_5 q | B \rangle = F_B, \quad \langle 0 | \bar{Q}_v \vec{\gamma} q | B^* \rangle = F_{B^*} \vec{e}, \quad (3.12)$$

and dots mean contribution of higher states. Therefore,

$$F_B = F_{B^*} = \sqrt{2} F. \quad (3.13)$$

The usual definition of the decay constants is

$$\langle 0 | \bar{Q}_v \gamma^\mu \gamma_5 q | B \rangle_r = i f_B p^\mu, \quad \langle 0 | \bar{Q}_v \gamma^\mu q | B^* \rangle_r = i m f_{B^*} e^\mu, \quad (3.14)$$

where the relativistic normalization

$${}_r \langle B, p' | B, p \rangle_r = (2\pi)^3 2p^0 \delta(\vec{p}' - \vec{p}) \quad (3.15)$$

of single-meson states is used (this normalization becomes meaningless in the limit $m \rightarrow \infty$, and therefore cannot be used in HQET). Comparing (3.14) for B or B^* at rest with (3.12), we arrive at

$$f_B = f_{B^*} = \frac{2F}{\sqrt{m}}, \quad (3.16)$$

up to $1/m$ corrections.

Let's consider the correlator (3.11) with $P = +1$. Equating the ground-state contributions to the spectral densities of the left-hand side and the right-hand one, we obtain

$$\langle 0 | j_2 | M \rangle \langle M | j_1^+ | 0 \rangle = F^2 \text{Tr} \Gamma_2 \frac{1 + \gamma^0}{2} \bar{\Gamma}_1. \quad (3.17)$$

Therefore,

$$\langle 0 | \bar{Q}_v \Gamma q | M \rangle = \frac{F}{\sqrt{2}} \text{Tr} \Gamma \frac{1 + \gamma^0}{2} \Gamma_M, \quad (3.18)$$

where the matrix

$$\Gamma_M = \begin{cases} -i\gamma_5 & \text{for } B \\ i\not{v} & \text{for } B^* \end{cases} \quad (3.19)$$

is defined up to a phase factor. We can re-write this result for the relativistic normalization of the meson state, in covariant notation:

$$\langle 0 | \bar{Q}_v \Gamma q | M \rangle_r = \sqrt{m} F \text{Tr} \Gamma \mathcal{M}, \quad (3.20)$$

where

$$\mathcal{M} = \frac{1 + \not{v}}{2} \begin{cases} -i\gamma_5 & \text{for } B \\ i\not{v} & \text{for } B^* \end{cases} \quad (3.21)$$

(of course, one can re-define the phases of $|M\rangle$ and hence of \mathcal{M}).

4 Quark–antiquark distribution amplitudes

After safely returning to the ideal world with a 0^+ heavy antiquark, we want to invent an operator which probes more details of the structure of B -meson than the local current (3.1). To this end, we consider a bilocal gauge-invariant operator

$$\tilde{O}(t) = Q_v^*(0)[0, z]q(z), \quad (4.1)$$

where

$$[x, y] = P \exp \left[-i \int_x^y A_\mu(z) dz^\mu \right], \quad (4.2)$$

and

$$z^2 = 0, \quad t = v \cdot z. \quad (4.3)$$

Its matrix element from the ground-state meson to vacuum has 2 Dirac structures:

$$\langle 0 | \tilde{O}(t) | M \rangle = F \left[\tilde{\varphi}_+(t) + \frac{\tilde{\varphi}_-(t) - \tilde{\varphi}_+(t)}{2t} \not{z} \right] u. \quad (4.4)$$

because $\not{p}u = u$.

In what follows, we shall often use light-front components of vectors. Let's introduce (in the v rest frame) two light-like vectors

$$\begin{aligned} n_{\pm}^{\mu} &= (1, \mp 1, \vec{0}), \\ n_+^2 &= n_-^2 = 0, \quad n_+ \cdot n_- = 2. \end{aligned} \quad (4.5)$$

Light-front components of any vector a are defined as

$$a_{\pm} = a \cdot n_{\pm} = a^0 \pm a^1. \quad (4.6)$$

We have

$$\begin{aligned} a^{\mu} &= \frac{1}{2} (a_+ n_-^{\mu} + a_- n_+^{\mu}) + a_{\perp}^{\mu}, \\ a \cdot b &= \frac{1}{2} (a_+ b_- + a_- b_+) - \vec{a}_{\perp} \cdot \vec{b}_{\perp}. \end{aligned} \quad (4.7)$$

In particular,

$$v^{\mu} = \frac{1}{2} (n_+^{\mu} + n_-^{\mu}), \quad v_+ = v_- = 1, \quad \vec{v}_{\perp} = \vec{0}. \quad (4.8)$$

We shall also use light-front components of γ^{μ} :

$$\gamma_{\pm} = \gamma \cdot n_{\pm} = \not{n}_{\pm}. \quad (4.9)$$

The definition (4.4) can be re-written as

$$\langle 0 | \tilde{O}(t) | M \rangle = \frac{1}{2} F [\tilde{\varphi}_+(t) \gamma_- + \tilde{\varphi}_-(t) \gamma_+] u. \quad (4.10)$$

If we introduce the operators

$$\tilde{O}_{\pm}(t) = \gamma_{\pm} \tilde{O}(t), \quad (4.11)$$

then

$$\langle 0 | \tilde{O}_{\pm}(t) | M \rangle = F \tilde{\varphi}_{\pm}(t) \gamma_{\pm} u. \quad (4.12)$$

The B -meson distribution amplitudes are the Fourier transforms of these functions:

$$\varphi_{\pm}(\omega) = \frac{1}{2\pi} \int \tilde{\varphi}_{\pm}(t) e^{i\omega t} dt, \quad \tilde{\varphi}_{\pm}(t) = \int \varphi_{\pm}(\omega) e^{-i\omega t} d\omega. \quad (4.13)$$

They are normalized by

$$\tilde{\varphi}_{\pm}(0) = \int_0^{\infty} \varphi_{\pm}(\omega) d\omega = 1. \quad (4.14)$$

The function $\varphi_+(\omega)$ is the leading-twist distribution amplitude, and $\varphi_-(\omega)$ – the subleading-twist one (though there is no good definition of twist in HQET). We can formally introduce the operators

$$\begin{aligned} O_{\pm}(\omega) &= \frac{1}{2\pi} \int \tilde{O}_{\pm}(t) e^{i\omega t} dt = Q_v^*(0) \gamma_{\pm} \delta(iD_+ - \omega) q(0), \\ \tilde{O}_{\pm}(t) &= \int O_{\pm}(\omega) e^{-i\omega t} d\omega, \end{aligned} \quad (4.15)$$

then

$$\langle 0 | O_{\pm}(\omega) | M \rangle = F \varphi_{\pm}(\omega) \gamma_{\pm} u. \quad (4.16)$$

The distribution amplitudes describe the distribution in the light-front component p_+ of the light-quark momentum in B -meson.

The expansion of the operators (4.11) in t reads

$$\begin{aligned} \tilde{O}_{\pm}(t) &= \sum_{n=0}^{\infty} O_{\pm}^{(n)} \frac{(-it)^n}{n!}, \\ O_{\pm}^{(n)} &= \int O_{\pm}(\omega) \omega^n d\omega = Q_v^* \gamma_{\pm} (iD_+)^n q, \end{aligned} \quad (4.17)$$

or for matrix elements

$$\begin{aligned} \tilde{\varphi}_{\pm}(t) &= \sum_{n=0}^{\infty} \langle \omega^n \rangle_{\pm} \frac{(-it)^n}{n!}, \\ \langle \omega^n \rangle_{\pm} &= \int_0^{\infty} \varphi_{\pm}(\omega) \omega^n d\omega, \\ \langle 0 | O_{\pm}^{(n)} | M \rangle &= F \langle \omega^n \rangle_{\pm} \gamma_{\pm} u. \end{aligned} \quad (4.18)$$

We can also reconstruct $O_{\pm}(\omega)$ from $O_{\pm}^{(n)}$:

$$O_{\pm}(\omega) = \int_{-i\infty}^{+i\infty} O_{\pm}^{(n)} \omega^{-n-1} \frac{dn}{2\pi i}, \quad (4.19)$$

or for matrix elements

$$\varphi_{\pm}(\omega) = \int_{-i\infty}^{+i\infty} \langle \omega^n \rangle_{\pm} \omega^{-n-1} \frac{dn}{2\pi i} \quad (4.20)$$

(you can easily check this by substituting (4.17) into (4.19); integration in dn yields $\delta(\log(\omega'/\omega))$).

The first moments $\langle \omega \rangle_{\pm}$ can be found from the equations of motion. The equation of motion for the heavy antiquark is

$$Q_v^* \overleftarrow{D}_0 = 0, \quad (4.21)$$

and therefore we obtain

$$\langle 0 | Q_v^* D_0 q | M \rangle = \langle 0 | \partial_0 (Q_v^* q) | M \rangle = -i F \bar{\Lambda} u. \quad (4.22)$$

The vector part has the structure

$$\langle 0 | Q_v^* \vec{D} q | M \rangle = a F \vec{\gamma} u. \quad (4.23)$$

The equation of motion of the light quark is

$$\langle 0 | Q_v^* \not{D} q | M \rangle = 0, \quad (4.24)$$

and using $\not{D} = D_0 \gamma^0 - \vec{D} \cdot \vec{\gamma}$ we obtain

$$\langle 0 | Q_v^* \vec{D} q | M \rangle = \frac{i}{3} F \bar{\Lambda} \vec{\gamma} u. \quad (4.25)$$

Finally, we arrive at

$$\langle \omega \rangle_+ = \frac{4}{3} \bar{\Lambda}, \quad \langle \omega \rangle_- = \frac{2}{3} \bar{\Lambda}. \quad (4.26)$$

Let's now consider the second moments. They involve two new (nonperturbative) hadronic parameters:

$$\langle 0 | Q_v^* \vec{E} \cdot \vec{\alpha} q | M \rangle = -i F \lambda_E^2 u, \quad \langle 0 | Q_v^* \vec{H} \cdot \vec{\sigma} q | M \rangle = -F \lambda_H^2 u, \quad (4.27)$$

where

$$\vec{E} = i[D_0, \vec{D}], \quad \vec{H} = i\vec{D} \times \vec{D}, \quad \vec{\alpha} = \gamma^0 \vec{\gamma}, \quad \vec{\sigma} = -\vec{\gamma} \gamma_5 \gamma^0. \quad (4.28)$$

Now we can calculate all matrix elements with 2 derivatives. From (4.22) and (4.25), using the heavy-antiquark equation of motion (4.21), we immediately find

$$\langle 0 | Q_v^* D_0^2 q | M \rangle = -F \bar{\Lambda}^2 u, \quad \langle 0 | Q_v^* D_0 \vec{D} q | M \rangle = \frac{1}{3} F \bar{\Lambda}^2 \vec{\gamma} u. \quad (4.29)$$

Using the definition (4.27) of λ_E^2 , we immediately find

$$\langle 0 | Q_v^* \vec{D} D_0 q | M \rangle = \frac{1}{3} F (\bar{\Lambda}^2 + \lambda_E^2) \vec{\gamma} u. \quad (4.30)$$

The second spatial derivatives have the structure

$$\langle 0 | Q_v^* D^i D^j q | M \rangle = F \left(b \delta^{ij} - \frac{i}{6} \lambda_H^2 \epsilon^{ijk} \sigma^k \right) u, \quad (4.31)$$

where the second coefficient follows from the definition (4.27) of λ_H^2 . We find b using the light-quark equation of motion (4.24):

$$\langle 0 | Q_v^* D^i D^j q | M \rangle = -\frac{1}{3} F \left[(\bar{\Lambda}^2 + \lambda_E^2 + \lambda_H^2) \delta^{ij} + \frac{i}{2} \lambda_H^2 \epsilon^{ijk} \sigma^k \right] u. \quad (4.32)$$

Finally, we arrive at

$$\langle \omega^2 \rangle_+ = 2\bar{\Lambda}^2 + \frac{2}{3} \lambda_E^2 + \frac{1}{3} \lambda_H^2, \quad \langle \omega^2 \rangle_- = \frac{2}{3} \bar{\Lambda}^2 + \frac{1}{3} \lambda_H^2. \quad (4.33)$$

What about B -meson distribution amplitudes in the real world with a $\frac{1}{2}^-$ heavy antiquark? It has 4 distribution amplitudes, as any pseudoscalar meson:

$$\begin{aligned} \langle 0 | Q_v^*(0) [0, z] \gamma_5 q(z) | B \rangle_r &= -i f_B m \tilde{\varphi}_P, \\ \langle 0 | Q_v^*(0) [0, z] \gamma^\mu \gamma_5 q(z) | B \rangle_r &= f_B [i \tilde{\varphi}_{A1} p^\mu - m \tilde{\varphi}_{A2} z^\mu], \\ \langle 0 | Q_v^*(0) [0, z] \sigma^{\mu\nu} \gamma_5 q(z) | B \rangle_r &= i f_B \tilde{\varphi}_T (p^\mu z^\nu - p^\nu z^\mu). \end{aligned} \quad (4.34)$$

Similarly to (3.20), we have

$$\langle 0 | Q_v^*(0) [0, z] \Gamma q(z) | M \rangle_r = F \text{Tr} \Gamma \left[\tilde{\varphi}_+ + \frac{\tilde{\varphi}_- - \tilde{\varphi}_+}{2t} \not{z} \right] \mathcal{M}. \quad (4.35)$$

Therefore these 4 QCD distribution amplitudes can be expressed via 2 HQET ones:

$$\begin{aligned} \tilde{\varphi}_P &= \frac{\tilde{\varphi}_+(t) + \tilde{\varphi}_-(t)}{2}, \quad \tilde{\varphi}_{A1} = \tilde{\varphi}_+(t), \\ \tilde{\varphi}_{A2} = \tilde{\varphi}_T &= \frac{i}{2} \frac{\tilde{\varphi}_+(t) - \tilde{\varphi}_-(t)}{t}. \end{aligned} \quad (4.36)$$

The QCD distribution amplitudes are usually considered as functions of the longitudinal momentum fraction $x = \omega/m$. It is usually assumed that $\varphi_{A1}(x) \sim x$ at $x \rightarrow 0$, and $\varphi_P(x) \rightarrow \text{const}$. Therefore, we shall assume that $\varphi_+(\omega) \sim \omega$ at $\omega \rightarrow 0$, and $\varphi_-(\omega) \rightarrow \text{const}$. The QCD distribution amplitudes are normalized as

$$\tilde{\varphi}_P(0) = \tilde{\varphi}_{A1}(0) = 1, \quad \tilde{\varphi}_{A2}(0) = \tilde{\varphi}_T(0) = \frac{\bar{\Lambda}}{3} \quad (4.37)$$

(to derive the last formula, we used the t expansion (4.18) and the first moments (4.26)).

The vector meson B^* is described by 6 distribution amplitudes in QCD. All 10 distribution amplitudes (4 for B plus 6 for B^*) are expressed via 2 HQET distribution amplitudes $\varphi_\pm(\omega)$. This is a consequence of the heavy-quark spin symmetry. We don't present formulae for B^* here; they can be found in [6].

5 Quark–antiquark–gluon distribution amplitudes

Now we shall discuss relations of quark–antiquark distribution amplitudes and quark–antiquark–gluon ones following from the equations of motion. In order to apply them, we need to differentiate with respect to the coordinates of the light quark and the heavy antiquark separately. Therefore, we go slightly off the light cone. The generalization of (4.4) to an arbitrary z^2 is

$$\langle 0 | Q_v^*(0) q(z) | M \rangle = F \left[\tilde{\varphi}_+(t, z^2) + \frac{\tilde{\varphi}_-(t, z^2) - \tilde{\varphi}_+(t, z^2)}{2t} \not{z} \right] u, \quad (5.1)$$

where $t = v \cdot z$, and the fixed-point gauge $x_\mu A^\mu(x) = 0$ is used to simplify notation.

In order to apply the light-quark equation of motion (4.24), we apply

$$\gamma^\mu \frac{\partial}{\partial z^\mu}$$

to this definition. The differentiation of the right-hand side is straightforward. In the left-hand side we write

$$\langle 0 | Q_v^*(0) \gamma^\mu (\partial_\mu - iA_\mu(z) + iA_\mu(z)) q(z) | M \rangle .$$

The first two terms yield zero. In the fixed-point gauge

$$A_\mu(z) = \int_0^1 G_{\nu\mu}(uz) u z^\nu du , \quad (5.2)$$

where $G_{\mu\nu} = gG_{\mu\nu}^a t^a$.

The matrix element of the operator containing $G_{\mu\nu}$ contains 4 structures:

$$\begin{aligned} \langle 0 | Q_v^*(0) [0, uz] iG_{\nu\mu}(uz) z^\nu [uz, z] q(z) | M \rangle = \\ - F \left[(v_\mu \not{z} - t\gamma_\mu)(\tilde{\psi}_A - \tilde{\psi}_V) + i\sigma_{\mu\nu} z^\nu \tilde{\psi}_V - z_\mu \tilde{\psi}_X + \frac{z_\mu}{t} \not{z} \tilde{\psi}_Y \right] u , \end{aligned} \quad (5.3)$$

and is parametrized by 4 quark-antiquark-gluon distribution amplitudes (note that this matrix elements vanishes when multiplied by z^μ).

Equating the coefficients of u and $\not{z}u$, we obtain two consequences of the light-quark equation of motion:

$$\tilde{\varphi}'_- + \frac{\tilde{\varphi}_- - \tilde{\varphi}_+}{t} = 2t \int_0^1 (\tilde{\psi}_A - \tilde{\psi}_V) u du , \quad (5.4)$$

$$\tilde{\varphi}'_+ - \tilde{\varphi}'_- + \frac{\tilde{\varphi}_- - \tilde{\varphi}_+}{t} + 4t \frac{\partial \tilde{\varphi}_+}{\partial z^2} = 2t \int_0^1 (2\tilde{\psi}_V + \tilde{\psi}_A + \tilde{\psi}_X) u du . \quad (5.5)$$

Similarly, to use the heavy-antiquark equation of motion we apply

$$Q_v^*(0) v^\mu \frac{\partial}{\partial z^\mu} q(z) = v^\mu \partial_\mu (Q_v^*(0) q(z)) - v^\mu Q_v^*(0) (\overleftarrow{\partial}_\mu + iA_\mu(0) - iA_\mu(0)) q(z) .$$

The first two terms in the last operator yield zero. Now we move the heavy antiquark; the center of the fixed-point gauge must be constant, and we have to use the gauge $(x - z)^\mu A_\mu(x) = 0$:

$$A_\mu(0) = - \int_0^1 G_{\nu\mu}(uz) (1 - u) z^\nu du . \quad (5.6)$$

We arrive at two consequences of the heavy-antiquark equation of motion:

$$\tilde{\varphi}'_+ + \frac{\tilde{\varphi}_- - \tilde{\varphi}_+}{2t} + i\bar{\Lambda} \tilde{\varphi}_+ + 2t \frac{\partial \tilde{\varphi}_+}{\partial z^2} = -t \int_0^1 (\tilde{\psi}_A + \tilde{\psi}_X) (1 - u) du , \quad (5.7)$$

$$\tilde{\varphi}'_- - \tilde{\varphi}'_+ + \frac{\tilde{\varphi}_+ - \tilde{\varphi}_-}{t} + i\bar{\Lambda} (\tilde{\varphi}_- - \tilde{\varphi}_+) + 2t \left(\frac{\partial \tilde{\varphi}_-}{\partial z^2} - \frac{\partial \tilde{\varphi}_+}{\partial z^2} \right) = 2t \int_0^1 (\tilde{\psi}_A + \tilde{\psi}_Y) (1 - u) du . \quad (5.8)$$

The functions

$$\left. \frac{\partial \tilde{\varphi}_{\pm}(t, z^2)}{\partial z^2} \right|_{z^2=0}$$

are some new (non-leading) quark–antiquark distribution amplitudes. We are not interested in them here. There are 2 equations involving only our familiar distribution amplitudes $\tilde{\varphi}_{\pm}(t, 0)$, namely, (5.4) and a combination of (5.5) and (5.7):

$$\begin{aligned} \tilde{\varphi}'_- + \frac{\tilde{\varphi}_- - \tilde{\varphi}_+}{t} &= 2t \int_0^1 (\tilde{\psi}_A - \tilde{\psi}_V) u \, du, \\ \tilde{\varphi}'_+ + \tilde{\varphi}'_- + 2i\bar{\Lambda}\tilde{\varphi}_+ &= -2t \int_0^1 (\tilde{\psi}_A + \tilde{\psi}_X + 2\tilde{\psi}_V u) du. \end{aligned} \quad (5.9)$$

The quark–antiquark–gluon distribution amplitudes in the momentum space are defined as

$$\tilde{\psi}_i(t, u) = \int \psi_i(\omega, \xi) e^{-i(\omega + \xi u)t} d\omega d\xi. \quad (5.10)$$

Performing Fourier transform of the equations (5.9), we obtain

$$\begin{aligned} \omega \frac{d\varphi_-(\omega)}{d\omega} + \varphi_+(\omega) &= I(\omega), \\ (\omega - 2\bar{\Lambda})\varphi_+(\omega) + \omega\varphi_-(\omega) &= J(\omega), \end{aligned} \quad (5.11)$$

where

$$\begin{aligned} I(\omega) &= 2 \frac{d}{d\omega} \int_0^\omega d\rho \int_{\omega-\rho}^\infty \frac{d\xi}{\xi} \frac{\partial}{\partial \xi} [\psi_A(\rho, \xi) - \psi_V(\rho, \xi)], \\ J(\omega) &= -2 \frac{d}{d\omega} \int_0^\omega d\rho \int_{\omega-\rho}^\infty \frac{d\xi}{\xi} [\psi_A(\rho, \xi) + \psi_X(\rho, \xi)] - 4 \int_0^\omega d\rho \int_{\omega-\rho}^\infty \frac{d\xi}{\xi} \frac{\partial}{\partial \xi} \psi_V(\rho, \xi). \end{aligned}$$

If we insist on having $\varphi_+(0) = 0$, then the condition

$$J(0) = -2 \int_0^\infty \frac{d\xi}{\xi} [\psi_A(0, \xi) + \psi_X(0, \xi)] = 0 \quad (5.12)$$

must be satisfied. It should follow from vanishing of $\psi_{A,X}(\omega, \xi)$ at $\omega \rightarrow 0$ (behaviour of the three-particle distribution amplitudes at the boundaries is discussed in [14]).

If we knew $\psi_{A,V,X}(\omega, \xi)$, we could solve the equations (5.11) for $\varphi_{\pm}(\omega)$. This is not very realistic, because we don't know them. The solution is a sum of two terms:

$$\varphi_{\pm}(\omega) = \varphi_{\pm}^{(WW)}(\omega) + \varphi_{\pm}^{(g)}(\omega), \quad (5.13)$$

where $\varphi_{\pm}^{(WW)}(\omega)$ is the solution of (5.11) in the case if all quark–antiquark–gluon distribution amplitudes vanish (it is called the Wandzura–Wilczek part of the solution), and $\varphi_{\pm}^{(g)}(\omega)$ is induced by the gluonic terms. The Wandzura–Wilczek part is

$$\varphi_+^{(WW)}(\omega) = \frac{\omega}{2\bar{\Lambda}^2} \theta(2\bar{\Lambda} - \omega), \quad \varphi_-^{(WW)}(\omega) = \frac{2\bar{\Lambda} - \omega}{2\bar{\Lambda}^2} \theta(2\bar{\Lambda} - \omega) \quad (5.14)$$

(Fig. 6); it satisfies the normalization conditions (4.14). Note that $2\bar{\Lambda}$ is the maximum value of p_+ of the light quark if we assume that B -meson (having the residual energy $\bar{\Lambda}$) consists of the on-shell heavy antiquark (always having zero residual energy) and the on-shell light quark. The gluon-induced part is given by some explicit integrals of $\psi_{A,V,X}(\omega, \xi)$; we don't present these long expressions here, they can be found in [8].

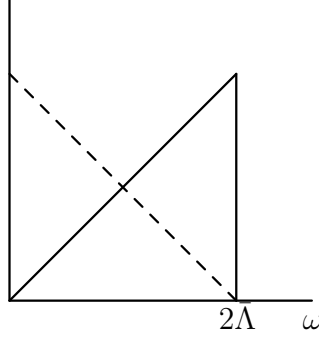


Figure 6: The Wandzura–Wilczek parts of $\varphi_+(\omega)$ (solid line) and $\varphi_-(\omega)$ (dashed line)

The moments (4.18) also consist of two contributions. The Wandzura–Wilczek parts are

$$\langle \omega^n \rangle_+^{(WW)} = \frac{2(2\bar{\Lambda})^n}{n+2}, \quad \langle \omega^n \rangle_-^{(WW)} = \frac{2(2\bar{\Lambda})^n}{(n+1)(n+2)}. \quad (5.15)$$

The gluon-induced part does not contribute to the zeroth moments (normalization) and the first ones; its contribution to the second moments (4.33) is expressed via the normalizations of the quark–antiquark–gluon distribution amplitudes:

$$\begin{aligned} \int \psi_A(\omega, \xi) d\omega d\xi &= \frac{1}{3} \lambda_E^2, \\ \int \psi_V(\omega, \xi) d\omega d\xi &= \frac{1}{3} \lambda_H^2, \\ \int \psi_X(\omega, \xi) d\omega d\xi &= 0. \end{aligned} \quad (5.16)$$

The contributions to the higher moments are expressed via the moments of $\psi_{A,V,X}(\omega, \xi)$; the explicit formulae can be found in [8].

6 Evolution

Until now, we neglected renormalization of the considered operators. In this section we shall discuss renormalization of the leading-twist B -meson distribution amplitude.

Let's summarize what we know about the bare operators of interest. There are 3 families of such operators, or 3 “representations” (t, ω, n) :

$$\begin{aligned}\tilde{O}_+(t) &= Q_v^*(0)\gamma_+[0, z]q(z), \\ O_+(\omega) &= Q_v^*(0)\gamma_+\delta(iD_+ - \omega)q(0), \\ O_+^{(n)} &= Q_v^*(0)\gamma_+(iD_+)^n q(0).\end{aligned}\tag{6.1}$$

They can be converted into each other:

$$\begin{aligned}\tilde{O}_+(t) &= \int O_+(\omega)e^{-i\omega t}d\omega = \sum_{n=0}^{\infty} O_+^{(n)} \frac{(-it)^n}{n!}, \\ O_+(\omega) &= \int \tilde{O}_+(t)e^{i\omega t} \frac{dt}{2\pi} = \int_{-i\infty}^{+i\infty} O_+^{(n)} \omega^{-n-1} \frac{dn}{2\pi i}, \\ O_+^{(n)} &= \left(i \frac{d}{dt} \right)^n O_+(t) \Big|_{t=0} = \int_0^{\infty} O_+(\omega) \omega^n d\omega.\end{aligned}\tag{6.2}$$

As we shall see, not all of these relations survive renormalization.

We shall calculate matrix elements of these bare operators, therefore, we need the Feynman rules for them. If we retain $i\partial_+$ in all brackets $(iD_+)^n$ in $O_+^{(n)}$, we obtain the quark–antiquark vertex with p_+^n shown in Fig. 7. When transformed to the ω -representation (6.2), this gives $\delta(p_+ - \omega)$, as expected from the form (6.1) of $O_+(\omega)$.

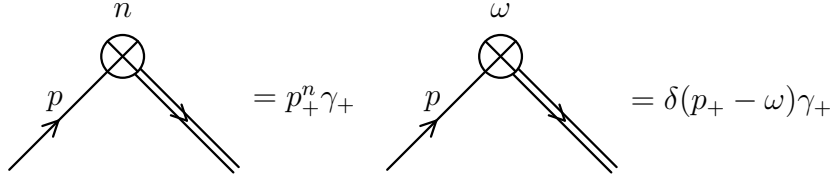


Figure 7: Quark–antiquark vertices

Now let's retain a single A_+ in $(iD_+)^n$. After some combinatorics, this gives

$$(i\partial_+ + A_+)^n \rightarrow \sum_{m=1}^n \binom{n}{m} [(i\partial_+)^{m-1} A_+] (i\partial_+)^{n-m}.\tag{6.3}$$

This gives the quark–antiquark–gluon vertex of $O_+^{(n)}$ with

$$\sum_{m=1}^n \binom{n}{m} k_+^{m-1} p_+^{n-m} = \frac{(p_+ + k_+)^n - p_+^n}{k_+},\tag{6.4}$$

shown in Fig. 8. Transforming it to the ω -representation (6.2) gives two δ -functions (Fig. 8). Of course, the integral in $d\omega$ of this vertex vanishes, because $O_+^{(0)}$ does not interact with gluons.

$$\begin{aligned}
& \text{Top diagram: } \text{Vertex} = \frac{(p_+ + k_+)^n - p_+^n}{k_+} g_0 t^a n_+^\mu \gamma_+ \\
& \text{Bottom diagram: } \text{Vertex} = \frac{\delta(p_+ + k_+ - \omega) - \delta(p_+ - \omega)}{k_+} g_0 t^a n_+^\mu \gamma_+
\end{aligned}$$

Figure 8: Quark-antiquark-gluon vertices

The bare operators $O_+(\omega)$ can be expressed via the renormalized operators $O_+(\omega; \mu)$ (μ is the $\overline{\text{MS}}$ renormalization scale), or vice versa:

$$O_+(\omega) = \int Z_+(\omega, \omega'; \mu) O_+(\omega'; \mu) d\omega', \quad (6.5)$$

$$O_+(\omega; \mu) = \int Z_+^{-1}(\omega, \omega'; \mu) O_+(\omega') d\omega'. \quad (6.6)$$

The renormalization “matrices” $Z_+(\omega, \omega'; \mu)$ and $Z_+^{-1}(\omega, \omega'; \mu)$ are inverse to each other:

$$\begin{aligned}
& \int Z_+(\omega, \omega''; \mu) Z_+^{-1}(\omega'', \omega'; \mu) d\omega'' = \delta(\omega - \omega'), \\
& \int Z_+^{-1}(\omega, \omega''; \mu) Z_+(\omega'', \omega'; \mu) d\omega'' = \delta(\omega - \omega').
\end{aligned} \quad (6.7)$$

The renormalized operators $O_+(\omega; \mu)$ obey the renormalization-group equation

$$\frac{\partial O_+(\omega; \mu)}{\partial \log \mu} + \int \Gamma_+(\omega, \omega'; \mu) O_+(\omega'; \mu) d\omega' = 0, \quad (6.8)$$

where the anomalous dimension “matrix” is

$$\begin{aligned}
\Gamma_+(\omega, \omega'; \mu) &= \int Z_+^{-1}(\omega, \omega''; \mu) \frac{\partial Z_+(\omega'', \omega'; \mu)}{\partial \log \mu} d\omega'' \\
&= - \int \frac{\partial Z_+^{-1}(\omega, \omega''; \mu)}{\partial \log \mu} Z_+(\omega'', \omega'; \mu) d\omega''.
\end{aligned} \quad (6.9)$$

This equation can be derived in two ways: either we differentiate (6.5) in $d \log \mu$ and obtain 0 (because the bare operator is μ -independent), or we differentiate (6.6).

At one loop

$$\begin{aligned}
Z_+(\omega, \omega'; \mu) &= \delta(\omega - \omega') + z_+^{(1)}(\omega, \omega'; \mu) a_s + \cdots \\
Z_+^{-1}(\omega, \omega'; \mu) &= \delta(\omega - \omega') - z_+^{(1)}(\omega, \omega'; \mu) a_s + \cdots \\
\Gamma_+(\omega, \omega'; \mu) &= \Gamma_+^{(1)}(\omega, \omega'; \mu) a_s + \cdots
\end{aligned} \tag{6.10}$$

where

$$a_s = \frac{\alpha_s(\mu)}{4\pi}.$$

From (6.9) we obtain

$$\Gamma_+^{(1)}(\omega, \omega'; \mu) = \frac{\partial z_+^{(1)}(\omega, \omega'; \mu)}{\partial \log \mu} - 2\varepsilon z_+^{(1)}(\omega, \omega'; \mu). \tag{6.11}$$

The matrix element of $O_+(\omega)$ between a state with the light quark with momentum p and the heavy antiquark with momentum p' (which are off-shell) and vacuum is

$$\begin{aligned}
M &= \langle 0 | O_+(\omega) | q(p), Q_v^*(p') \rangle = Z_q^{1/2} \tilde{Z}_Q^{1/2} [\delta(p_+ - \omega) \gamma_+ + M_1 + M_2 + M_3] \\
&= \langle 0 | O_+(\omega; \mu) | q(p), Q_v^*(p') \rangle + a_s \int z_+^{(1)}(\omega, \omega'; \mu) \delta(p_+ - \omega') \gamma_+ d\omega',
\end{aligned} \tag{6.12}$$

where $M_{1,2,3}$ are the contributions of the one-loop diagrams shown in Fig. 9, the matrix element of the renormalized operator is finite at $\varepsilon \rightarrow 0$, and $z_+^{(1)}(\omega, \omega'; \mu)$ contains only negative powers of ε (in the term with $z_+^{(1)}$, we may substitute $\delta(p_+ - \omega') \gamma_+$ instead of $\langle 0 | O_+(\omega'; \mu) | q(p), Q_v^*(p') \rangle$). This allows us to find the renormalization “matrix” with one-loop accuracy.

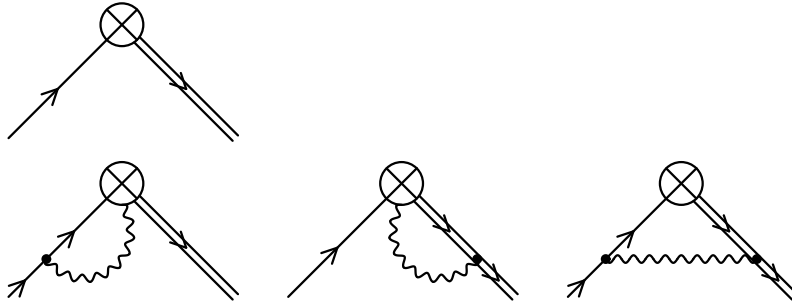


Figure 9: Diagrams for the matrix element of $O_+(\omega)$ between a quark–antiquark state and vacuum

Let’s calculate the first diagram (Fig. 10) for

$$p_+ = \omega', \quad p_\perp = 0, \quad p^2 = p_+ p_- < 0.$$

It is

$$M_1 = iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\delta(p_+ - \omega) - \delta(k_+ - \omega)}{p_+ - k_+} \frac{\gamma_+ \not{k} \gamma_+}{[-(p-k)^2 - i0] [-k^2 - i0]} . \quad (6.13)$$

The numerator can be simplified as

$$\gamma_+ \not{k} \gamma_+ = 2k_+ \gamma_+ .$$

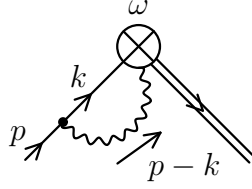


Figure 10: The first diagram

This diagram can be written in the form

$$M_1 = 2C_F \frac{g_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \gamma_+ \left[f_1(\omega, \omega') - \delta(\omega - \omega') \int f_1(\omega'', \omega') d\omega'' \right] , \quad (6.14)$$

where the term with $f_1(\omega, \omega')$ comes from the second δ -function in (6.13). This function is

$$\pi^{d/2} (-p^2)^{-\varepsilon} \Gamma(\varepsilon) f_1(\omega, \omega') = -i \frac{\omega}{\omega' - \omega} \int d^d k \frac{\delta(k_+ - \omega)}{[-(p-k)^2 - i0] [-k^2 - i0]} .$$

Now we use α parametrization

$$\frac{1}{-k^2 - i0} = \int_0^\infty e^{(k^2 + i0)\alpha} d\alpha \quad (6.15)$$

for both denominators, and also write the δ -function as

$$\delta(k_+ - \omega) = 2\delta(2(k_+ - \omega)) = 2 \int_{-\infty}^{+\infty} \exp[2(k_+ - \omega)\nu] \frac{d\nu}{2\pi} . \quad (6.16)$$

We obtain

$$\begin{aligned} \pi^{d/2} (-p^2)^{-\varepsilon} \Gamma(\varepsilon) f_1(\omega, \omega') &= -2i \frac{\omega}{\omega' - \omega} \int d^d k d\alpha_1 d\alpha_2 \frac{d\nu}{2\pi} \\ &\times \exp[\alpha_1 (k-p)^2 + \alpha_2 k^2 + 2i\nu(k \cdot n_+ - \omega)] . \end{aligned}$$

Shifting the integration momentum as

$$k' = k - \frac{\alpha_1 p - i\nu n_+}{\alpha_1 + \alpha_2}$$

to form a full square, we obtain

$$-2i \frac{\omega}{\omega' - \omega} \int d\alpha_1 d\alpha_2 \exp \left[\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} p^2 \right] \\ \times \int \frac{d\nu}{2\pi} \exp \left[2i\nu \left(\frac{\alpha_1}{\alpha_1 + \alpha_2} \omega' - \omega \right) \right] \int d^d k' \exp [(\alpha_1 + \alpha_2)(k'^2 + i0)]$$

The integral in $d^d k'$ is calculated using the Wick rotation $k_0 = ik_{E0}$:

$$\int d^d k e^{\alpha(k^2 + i0)} = i \int d^d k_E e^{-\alpha k_E^2} = i \left(\frac{\pi}{\alpha} \right)^{d/2}. \quad (6.17)$$

The integral in $d\nu$ gives a δ -function. We obtain an integral in two α -parameters:

$$(-p^2)^{-\varepsilon} \Gamma(\varepsilon) f_1(\omega, \omega') \\ = \frac{\omega}{\omega' - \omega} \int d\alpha_1 d\alpha_2 (\alpha_1 + \alpha_2)^{-d/2} \exp \left[\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} p^2 \right] \delta \left(\frac{\alpha_1}{\alpha_1 + \alpha_2} \omega' - \omega \right).$$

It is always possible to calculate one integral in a “radial” variable η in the space of α -parameters via Γ -function. In this case, the most convenient choice of such a variable is $\eta = \alpha_1 + \alpha_2$. Therefore, we insert $\delta(\alpha_1 + \alpha_2 - \eta) d\eta$ under the integral sign, and substitute $\alpha_i = \eta x_i$:

$$\frac{\omega}{\omega' - \omega} \int dx_1 dx_2 \delta(x_1 + x_2 - 1) \delta(x_1 \omega' - \omega) \int d\eta \eta^{-1+\varepsilon} e^{-(p^2)x_1 x_2 \eta}.$$

The final result is

$$f_1(\omega, \omega') = \frac{\theta(\omega' - \omega)}{(\omega' - \omega)^{1+\varepsilon}} \frac{\omega^{1-\varepsilon}}{(\omega')^{1-2\varepsilon}}. \quad (6.18)$$

Functions $F(\omega, \omega')$ which appear in the evolution kernel should be understood as distributions: they are always integrated with smooth test functions $\varphi(\omega')$. The distribution $[F(\omega, \omega')]_+$ is defined by

$$\int [F(\omega, \omega')]_+ \varphi(\omega') d\omega' = \int F(\omega, \omega') (\varphi(\omega') - \varphi(\omega)) d\omega'. \quad (6.19)$$

Therefore, formally we can write

$$F(\omega, \omega') = [F(\omega, \omega')]_+ + \delta(\omega - \omega') \int F(\omega, \omega'') d\omega''. \quad (6.20)$$

The result (6.14), (6.18) of the calculation of the diagram in Fig. 10 can be written via a +-distribution as

$$M_1 = 2C_F \frac{g_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \gamma_+ \left[[f_1(\omega, \omega')]_+ + \delta(\omega - \omega') \left(\int f_1(\omega, \omega'') d\omega'' - \int f_1(\omega'', \omega) d\omega'' \right) \right]. \quad (6.21)$$

The coefficient of $\delta(\omega - \omega')$ here can be calculated by substitution $x = \omega''/\omega$:

$$\int_1^\infty x^{-1+2\varepsilon}(1-x)^{-1-\varepsilon}dx - \int_0^1 x^{1-\varepsilon}(1-x)^{-1-\varepsilon}dx.$$

Substituting $x \rightarrow 1/x$ in the first integral, we have

$$\int_0^1 (x^{-\varepsilon} - x^{1-\varepsilon})(1-x)^{-1-\varepsilon}dx = \int_0^1 x^{-\varepsilon}(1-x)^{-\varepsilon}dx.$$

Therefore, the final result for M_1 (Fig. 10) is

$$M_1 = 2C_F \frac{g_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \gamma_+ \left[\left(\frac{\theta(\omega' - \omega)}{(\omega' - \omega)^{1+\varepsilon}} \frac{\omega^{1-\varepsilon}}{(\omega')^{1-2\varepsilon}} \right)_+ + \frac{\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)} \delta(\omega - \omega') \right]. \quad (6.22)$$

Now we shall calculate the second diagram (Fig. 11) for

$$p' \cdot v = \omega_1 < 0.$$

It is

$$M_2 = -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\delta(p_+ + k_+ - \omega) - \delta(p_+ - \omega)}{k_+} \frac{v_+ \gamma_+}{[-k^2 - i0] [-(p' - k) \cdot v - i0]}. \quad (6.23)$$

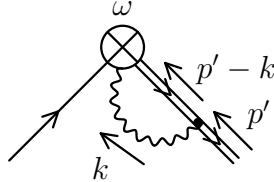


Figure 11: The second diagram

This diagram can be written in the form

$$M_2 = 2C_F \frac{g_0^2}{(4\pi)^{d/2}} \Gamma(\varepsilon) \gamma_+ \left[f_2(\omega - \omega') - \delta(\omega - \omega') \int f_2(\omega'') d\omega'' \right], \quad (6.24)$$

where the term with $f_2(\omega - \omega')$ comes from the first δ -function in (6.23). This function is

$$\pi^{d/2} \Gamma(\varepsilon) f_2(\omega'') = -\frac{i}{2\omega''} \int d^d k \frac{\delta(k_+ - \omega'')}{[-k^2 - i0] [-(p' - k) \cdot v - i0]}.$$

Now we use α -parametrization (6.15) for both denominators (it is convenient to multiply the linear denominator by 2 before this) and (6.16) for the δ -function, and obtain

$$-\frac{2i}{\omega''} \int d^d k d\alpha_1 d\alpha_2 \frac{d\nu}{2\pi} \exp [\alpha_1 k^2 + 2\alpha_2 (p' - k) \cdot v + 2i\nu (k \cdot n_+ - \omega'')] .$$

Shifting the integration momentum as

$$k' = k - \frac{\alpha_2 v - i\nu n_+}{\alpha_1}$$

to form a full square, we obtain

$$\begin{aligned} & -\frac{2i}{\omega''} \int d\alpha_1 d\alpha_2 \exp \left[-\frac{\alpha_2^2}{\alpha_1} + 2\omega_1 \alpha_2 \right] \\ & \times \int \frac{d\nu}{2\pi} \exp \left[2i\nu \left(\frac{\alpha_2}{\alpha_1} - \omega'' \right) \right] \int d^d k' \exp [\alpha_1 (k'^2 + i0)] . \end{aligned}$$

Taking the integrals in $d^d k'$ (6.17) and $d\nu$, we obtain the integral in two α -parameters:

$$\Gamma(\varepsilon) f_2(\omega'') = \frac{1}{\omega''} \int d\alpha_1 d\alpha_2 \alpha_1^{-d/2} \exp \left[-\frac{\alpha_2^2}{\alpha_1} + 2\omega_1 \alpha_2 \right] \delta \left(\frac{\alpha_2}{\alpha_1} - \omega'' \right) .$$

Now the best choice of the “radial” variable is α_1 , so we simply substitute $\alpha_2 = \alpha_1 y$:

$$\frac{1}{\omega''} \int dy \delta(y - \omega'') \int d\alpha_1 \alpha_1^{-1+\varepsilon} e^{-y(y-2\omega_1)\alpha_1} .$$

Finally,

$$f_2(\omega'') = \frac{\theta(\omega'')}{(\omega'')^{1+\varepsilon} (\omega'' - 2\omega_1)^\varepsilon} . \quad (6.25)$$

Now we rewrite M_2 (6.24), (6.25) via a $+$ -distribution:

$$M_2 = 2C_F \frac{g_0^2}{(4\pi)^{d/2}} \Gamma(\varepsilon) \gamma_+ \left[[f_2(\omega - \omega')]_+ + \delta(\omega - \omega') \left(\int_0^\omega f_2(\omega - \omega'') d\omega'' - \int_0^\infty f_2(\omega'') d\omega'' \right) \right] . \quad (6.26)$$

The coefficient of $\delta(\omega - \omega')$ is

$$- \int_\omega^\infty f_2(\omega'') d\omega'' . \quad (6.27)$$

We introduced ω_1 only to regularize possible infrared problems; we only need the UV divergence of this diagram, which does not depend on ω_1 . Therefore, we may assume $|\omega_1| \ll \omega$. The coefficient of $\delta(\omega - \omega')$ is

$$- \int_\omega^\infty (\omega'')^{-1-2\varepsilon} d\omega'' = -\frac{\omega^{-2\varepsilon}}{2\varepsilon} , \quad (6.28)$$

and the final result for M_2 (Fig. 11) is

$$M_2 = 2C_F \frac{g_0^2}{(4\pi)^{d/2}} \Gamma(\varepsilon) \gamma_+ \left[\left(\frac{\theta(\omega - \omega')}{(\omega - \omega')^{1+2\varepsilon}} \right)_+ - \frac{\omega^{-2\varepsilon}}{2\varepsilon} \delta(\omega - \omega') \right] . \quad (6.29)$$

This one-loop diagram contains a $1/\varepsilon^2$ UV divergence! It is in the coefficient of $\delta(\omega - \omega')$ given by the integral (6.27). The function $f_2(\omega'')$ has a $1/\varepsilon$ UV divergence in the integration

in transverse momentum; the longitudinal integral (6.27) is again UV divergent (6.28). The operator at the vertex of Fig. 11 is (after integration in ω) a light-like Wilson line; the heavy-antiquark propagator is a time-like Wilson line. The vertex correction to a time-like – light-like cusp on a Wilson line is known to have a $1/\varepsilon^2$ UV divergence at one loop [15].

The third diagram (Fig. 12) is

$$M_3 = -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\delta(k_+ - \omega) \gamma_+ \not{k} \not{p}}{[-(k-p)^2 - i0] [-k^2 - i0] [-(p' + p - k) \cdot v - i0]}. \quad (6.30)$$

The numerator can be simplified as

$$\gamma_+ \not{k} \not{p} = k_+ \gamma_+.$$

We shall now demonstrate that this diagram is UV-finite and hence does not contribute to the renormalization constant.

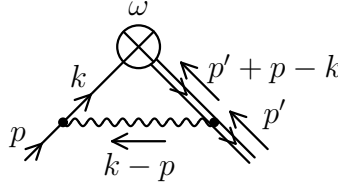


Figure 12: The third diagram

Using α -parametrization (6.15) for the denominators and (6.16) for the δ -function, and obtain

$$M_3 = -4iC_F g_0^2 \omega \gamma_+ \int d^d k d\alpha_1 d\alpha_2 d\alpha_3 \frac{d\nu}{2\pi} \times \exp \left[\alpha_1 (k-p)^2 + \alpha_2 k^2 + 2\alpha_3 (p+p'-k) \cdot v + 2i\nu (k \cdot n_+ - \omega) \right].$$

Shifting the integration momentum as

$$k' = k - \frac{\alpha_1 p + \alpha_3 v - i\nu n_+}{\alpha_1 + \alpha_2},$$

we obtain

$$\begin{aligned} M_3 &= -4iC_F g_0^2 \omega \gamma_+ \int d\alpha_1 d\alpha_2 d\alpha_3 e^{-A} \\ &\quad \times \int \frac{d\nu}{2\pi} \exp \left[2i\nu \left(\frac{\alpha_1 \omega' + \alpha_3}{\alpha_1 + \alpha_2} - \omega \right) \right] \int \frac{d^d k'}{(2\pi)^d} e^{(\alpha_1 + \alpha_2)(k'^2 + i0)} \\ &= 2C_F \frac{g_0^2}{(4\pi)^{d/2}} \omega \gamma_+ \int d\alpha_1 d\alpha_2 d\alpha_3 \delta \left(\frac{\alpha_1 \omega' + \alpha_3}{\alpha_1 + \alpha_2} - \omega \right) e^{-A}, \end{aligned}$$

where

$$A = \frac{\alpha_3(\alpha_3 - \alpha_2\omega') + \alpha_2(\alpha_1\omega' + \alpha_3)(-p^2)}{\alpha_1 + \alpha_2} + \alpha_3(-2\omega).$$

Inserting $\delta(\alpha_1 + \alpha_2 - \eta)d\eta$ under the integral sign and making substitutions $\alpha_{1,2} = \eta x_{1,2}$, $\alpha_3 = \eta y$, we have $A = a\eta$, and the integral in the “radial variable” η is easily calculated:

$$\begin{aligned} M_3 &= 2C_F \frac{g_0^2}{(4\pi)^{d/2}} \omega \gamma_+ \int dx_1 dy \delta(y + \omega' x_1 - \omega) \int d\eta \eta^\varepsilon e^{-a\eta} \\ &= 2C_F \frac{g_0^2}{(4\pi)^{d/2}} \Gamma(1 + \varepsilon) \frac{\omega}{\omega'} \gamma_+ \int_{\max(0, \omega - \omega')}^\omega \frac{dy}{a^{1+\varepsilon}}, \end{aligned}$$

where

$$a = \left[\omega - \omega' + \frac{\omega}{\omega'^2}(-p^2) - 2\omega_1 \right] y + (\omega - \omega') \frac{\omega}{\omega'^2} p^2.$$

The integral in y is finite at $\varepsilon \rightarrow 0$ (and easy to calculate).

Now we are ready to find the renormalization constant $Z_+(\omega, \omega'; \mu)$. Re-expressing

$$\frac{g_0^2}{(4\pi)^{d/2}} = a_s \mu^{2\varepsilon} e^{\gamma_E \varepsilon}$$

in the matrix element (6.12), we have

$$\begin{aligned} M &= \gamma_+ \left(Z_q \tilde{Z}_Q \right)^{1/2} \left[\delta(\omega - \omega') + 2C_F \frac{\alpha_s(\mu)}{4\pi\varepsilon} \right. \\ &\quad \times \left(\left(\frac{\theta(\omega' - \omega)}{\omega' - \omega} \frac{\omega}{\omega'} \right)_+ + \delta(\omega - \omega') \right. \\ &\quad \left. \left. + \left(\frac{\theta(\omega - \omega')}{\omega - \omega'} \right)_+ - \left(\frac{1}{2\varepsilon} - \log \frac{\omega}{\mu} \right) \delta(\omega - \omega') + \mathcal{O}(\varepsilon) \right) \right]. \end{aligned} \quad (6.31)$$

The quark-field renormalization constants in QCD and HQET th the Feynman gauge are

$$Z_q = 1 - C_F \frac{\alpha_s}{4\pi\varepsilon}, \quad \tilde{Z}_Q = 1 + 2C_F \frac{\alpha_s}{4\pi\varepsilon}. \quad (6.32)$$

Finally, the renormalization constant “matrix” at one loop is

$$\begin{aligned} Z_+(\omega, \omega'; \mu) &= \delta(\omega - \omega') + 2C_F \frac{\alpha_s(\mu)}{4\pi\varepsilon} \\ &\quad \times \left[\left(\frac{\theta(\omega' - \omega)}{\omega' - \omega} \frac{\omega}{\omega'} + \frac{\theta(\omega - \omega')}{\omega - \omega'} \right)_+ + \left(-\frac{1}{2\varepsilon} + \log \frac{\omega}{\mu} + \frac{5}{4} \right) \delta(\omega - \omega') \right]. \end{aligned} \quad (6.33)$$

The evolution kernel (6.9) has the structure

$$\Gamma_+(\omega, \omega'; \mu) = \Gamma(\omega, \omega'; a_s) + \left[-\Gamma(\omega, \omega'; a_s) \log \frac{\omega}{\mu} + \tilde{\gamma}_j(a_s) + \gamma(a_s) \right] \delta(\omega - \omega'). \quad (6.34)$$

The logarithm $\log(\omega/\mu)$ only appears linearly, to all orders of perturbation theory; the coefficient of this logarithm is the cusp anomalous dimension [15]:

$$\Gamma(a_s) = \Gamma_0 a_s + \Gamma_1 a_s^2 + \cdots \quad \Gamma_0 = 4C_F \quad (6.35)$$

(the two-loop term is also known [15]). The non-logarithmic part is written as $\tilde{\gamma}_j + \gamma$, where the anomalous dimension of the local current j (3.1) is

$$\tilde{\gamma}_j(a_s) = \tilde{\gamma}_{j0} a_s + \tilde{\gamma}_{j1} a_s^2 + \cdots \quad \tilde{\gamma}_{j0} = -3C_F \quad (6.36)$$

(the two- and three-loop terms are also known [16, 17]). It determines the evolution of $F(\mu)$; the difference

$$\gamma(a_s) = \gamma_0 a_s + \gamma_1 a_s^2 + \cdots \quad \gamma_0 = -2C_F \quad (6.37)$$

appears in the evolution equation for $\varphi_+(\omega; \mu)$. The non- δ term is

$$\Gamma(\omega, \omega', a_s) = \Gamma_0(\omega, \omega') a_s + \cdots \quad \Gamma_0(\omega, \omega') = \left(\frac{\theta(\omega' - \omega)}{\omega' - \omega} \frac{\omega}{\omega'} + \frac{\theta(\omega - \omega')}{\omega - \omega'} \right)_+ . \quad (6.38)$$

The evolution kernel has dimensionality of 1/energy.

The evolution equation for the distribution amplitude is

$$\frac{\partial \varphi_+(\omega; \mu)}{\partial \log \mu} + \left[-\Gamma(a_s) \log \frac{\omega}{\mu} + \gamma(a_s) \right] \varphi_+(\omega; \mu) + \int \Gamma(\omega, \omega'; a_s) \varphi_+(\omega'; \mu) d\omega' = 0 . \quad (6.39)$$

How to solve it? Powers ω^n are eigenfunctions of the integral operator in (6.39), by dimensionality:

$$\int \Gamma(\omega, \omega'; a_s) \omega'^n d\omega' = \tilde{\Gamma}(n, a_s) \omega^n , \quad (6.40)$$

where

$$\tilde{\Gamma}(n, a_s) = \tilde{\Gamma}_0 a_s + \cdots \quad \tilde{\Gamma}_0(n) = 4C_F [\psi(1+n) + \psi(1-n) + 2\gamma_E] , \quad (6.41)$$

from (6.38) (here γ_E is the Euler constant). They are not eigenfunctions of the whole evolution operator, due to the logarithmic term. We can construct solutions with the power depending on μ :

$$\frac{\partial}{\partial \log \mu} \left(\frac{\omega}{\mu_0} \right)^{n+\xi(\mu)} = \left(\frac{\omega}{\mu_0} \right)^{n+\xi(\mu)} \frac{d\xi}{d \log \mu} \log \frac{\omega}{\mu_0} .$$

If the function $\xi(\mu)$ obeys

$$\frac{d\xi}{d \log \mu} = \Gamma(a_s) , \quad (6.42)$$

then $\log \omega$ will cancel in the evolution equation.

Dividing the definition (6.42) by

$$\frac{d \log a_s}{d \log \mu} = -2\beta(a_s) , \quad \beta(a_s) = \beta_0 a_s + \beta_1 a_s^2 + \cdots \quad (6.43)$$

and integrating, we obtain

$$\xi = - \int_{a_{s0}}^{a_s} \frac{\Gamma(a_s)}{2\beta(a_s)} \frac{da_s}{a_s} = - \frac{\Gamma_0}{2\beta_0} \left[\log \frac{a_s}{a_{s0}} + \left(\frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0} \right) (a_s - a_{s0}) + \dots \right], \quad (6.44)$$

where $a_{s0} = \alpha_s(\mu_0)/(4\pi)$. We shall use ξ as an independent variable (“time”) in the evolution equation instead of μ . The function

$$\varphi(\omega, \xi) = \left(\frac{\omega}{\mu_0} \right)^{n+\xi} e^{U(\xi)} \quad (6.45)$$

satisfies the evolution equation

$$\frac{\partial \log \varphi}{\partial \xi} - \log \frac{\omega}{\mu_0} + \frac{\tilde{\Gamma}(n + \xi, a_s) + \gamma(a_s)}{\Gamma(a_s)} = 0 \quad (6.46)$$

if

$$\frac{dU}{d\xi} = - \log \frac{\mu}{\mu_0} - \frac{\tilde{\Gamma}(n + \xi, a_s) + \gamma(a_s)}{\Gamma(a_s)}.$$

At the leading order,

$$a_s = a_{s0} \exp \left(- \frac{2\beta_0}{\Gamma_0} \xi \right), \quad \log \frac{\mu}{\mu_0} = \frac{2\pi}{\beta_0} \left(\frac{1}{a_s} - \frac{1}{a_{s0}} \right) = \frac{2\pi}{\beta_0 a_{s0}} \left[\exp \left(\frac{2\beta_0}{\Gamma_0} \xi \right) - 1 \right],$$

and

$$U(\xi) = - \frac{2\pi}{\beta_0 a_{s0}} \left[\frac{\Gamma_0}{2\beta_0} \left(\exp \left(\frac{2\beta_0}{\Gamma_0} \xi \right) - 1 \right) - \xi \right] - \int_0^\xi \frac{\tilde{\Gamma}_0(n + \xi)}{\Gamma_0} d\xi - \frac{\gamma_0}{\Gamma_0} \xi.$$

Here, from (6.41),

$$\int_0^\xi \frac{\tilde{\Gamma}_0(n + \xi)}{\Gamma_0} d\xi = \log \frac{\Gamma(1 + n + \xi)\Gamma(1 - n)}{\Gamma(1 - n - \xi)\Gamma(1 + n)} + 2\gamma_E \xi.$$

Finally, at the leading order,

$$e^{U(\xi)} = \left(\frac{\Lambda_{\overline{\text{MS}}}}{\mu_0} \right)^{\frac{\Gamma_0}{2\beta_0} [\exp(\frac{2\beta_0}{\Gamma_0} \xi) - 1] - \xi} \frac{\Gamma(1 - n - \xi)\Gamma(1 + n)}{\Gamma(1 + n + \xi)\Gamma(1 - n)} \exp \left[- \left(\frac{\gamma_0}{\Gamma_0} + 2\gamma_E \right) \xi \right]. \quad (6.47)$$

The distribution amplitude at μ_0 can be expressed via its moments (4.20):

$$\varphi_+(\omega; \mu_0) = \int_{-i\infty}^{+i\infty} \langle \omega^{-1-n} \rangle_+^{(\mu_0)} \omega^n \frac{dn}{2\pi i}.$$

Substituting the solutions (6.45) instead of ω^n gives us the solution of the evolution equation (6.39) with initial conditions. At the leading order (6.47),

$$\begin{aligned} \varphi_+(\omega; \mu) &= \left(\frac{\Lambda_{\overline{\text{MS}}}}{\mu_0} \right)^{\frac{\Gamma_0}{2\beta_0} [\exp(\frac{2\beta_0}{\Gamma_0} \xi) - 1]} \exp \left[- \left(\frac{\gamma_0}{\Gamma_0} + 2\gamma_E \right) \xi \right] \left(\frac{\omega}{\Lambda_{\overline{\text{MS}}}} \right)^\xi \\ &\times \int_{-\infty}^{+\infty} \langle \omega^{-1-in} \rangle_+^{(\mu_0)} \omega^{in} \frac{\Gamma(1 - n - \xi)\Gamma(1 + n)}{\Gamma(1 + n + \xi)\Gamma(1 - n)} \frac{dn}{2\pi}. \end{aligned} \quad (6.48)$$

Qualitatively, each “bin” in the distribution amplitude at μ_0 near some finite ω' produces, after evolution to a larger μ , a radiative tail, slowly decreasing (as $1/\omega$) at $\omega \gg \omega'$, due to the evolution kernel $\Gamma(\omega, \omega'; a_s)$ (which behaves as $1/\omega$ at large ω , by dimensionality). Therefore, the behaviour of the distribution amplitude at large ω is $1/\omega$, up to logarithms. The normalization integral (4.14) of the distribution amplitude logarithmically diverges at large ω ; its moments (4.18) with $n > 0$ are power-divergent. This means that the expression (4.17) for the local operators $O_+^{(n)}$ ($n \geq 0$) via $O_+(\omega)$ is not valid for the renormalized operators. Renormalization of $O_+(\omega)$ removes UV divergences in transverse momenta; in order to renormalize $O_+^{(n)}$, we should also remove longitudinal UV divergences (at large ω).

As an illustration [7], let's suppose that at a low μ_0 the distribution amplitude $\varphi_+(\omega)$ is given by the simple model (7.24). Namely, this initial condition is taken at the scale μ_0 where

$$\alpha_s(\mu_0) = 1 \quad \Rightarrow \quad \frac{\mu_0}{\Lambda_{\overline{\text{MS}}}} = \exp \frac{2\pi}{\beta_0},$$

where the quark model is supposed to work, so that the light quark has only momenta of order $\bar{\Lambda}$ (no radiative tail). This function is shown by the solid line in Fig. (13), where ω is measured in units of ω_0 (7.25). Using the leading-order solution (6.48) of the evolution equation (6.39) and supposing $\bar{\Lambda}/\Lambda_{\overline{\text{MS}}} = 1.25$, we obtain the distribution amplitude at the scale where $\alpha_s(\mu) = 0.5$ (dashed line) and 0.3 (dashed-dotted line). We can see that the main part of the distribution amplitude (at $\omega \sim \bar{\Lambda}$) becomes lower and the radiative tail becomes more prominent when μ grows.

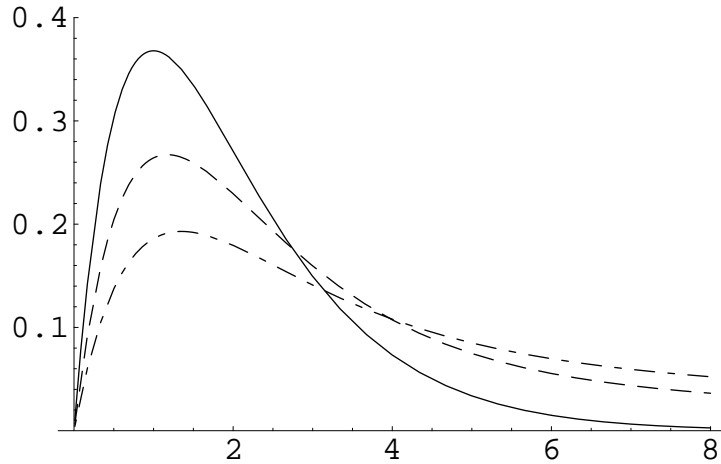


Figure 13: Evolution of B -meson distribution amplitude $\varphi_+(\omega; \mu)$

7 Sum rules

In order to estimate the quark–antiquark distribution amplitudes $\varphi_{\pm}(\omega)$ of B -meson from QCD sum rules, we consider the correlator of the local current j_+ (3.2) (having the quantum numbers of the ground-state meson) and the bilocal operator $\tilde{O}_{\pm}(t)$ (4.11):

$$i\langle T\tilde{O}_{\pm}(t)\bar{j}_+(-x)\rangle = \gamma_{\pm}\frac{1+\gamma^0}{2}\delta(\vec{x})\theta(x^0)\tilde{\Pi}_{\pm}(x^0, t). \quad (7.1)$$

We are most interested in its Fourier transform

$$\Pi_{\pm}(x^0, \omega) = \int \tilde{\Pi}_{\pm}(x^0, t)e^{i\omega t}\frac{dt}{2\pi}. \quad (7.2)$$

Analytically continuing it from $x^0 > 0$ to $x^0 = -i\tau$, we obtain

$$\Pi_{\pm}(\tau, \omega) = \int \rho_{\pm}(\varepsilon, \omega)e^{-\varepsilon\tau}d\varepsilon = F^2\tilde{\varphi}_{\pm}(\omega)e^{-\bar{\Lambda}\tau} + \Pi_{\pm}^c(\tau, \omega), \quad (7.3)$$

where $\rho_{\pm}(\varepsilon, \omega)$ is the spectral density. The correlator contains the contribution of the ground-state meson (written explicitly in (7.3)) and of the continuum of excited states.

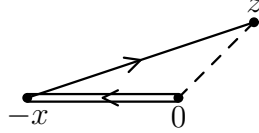


Figure 14: Correlator of the local and the bilocal operators

For sufficiently small τ , we can calculate this correlator theoretically. The perturbative contribution (Fig. 14) is given in the fixed-point gauge $x^{\mu}A_{\mu}(x) = 0$ by the light-quark propagator from $-x$ to z :

$$\tilde{\Pi}_{+}^{(1)}(\tau, t) = \frac{N_c}{2\pi^2\tau(\tau + 2it)^2}, \quad \tilde{\Pi}_{-}^{(1)}(\tau, t) = \frac{N_c}{2\pi^2\tau^2(\tau + 2it)}. \quad (7.4)$$

Its Fourier transform is

$$\Pi_{+}^{(1)}(\tau, \omega) = \frac{N_c}{8\pi^2\tau}\omega e^{-\omega\tau/2}, \quad \Pi_{-}^{(1)}(\tau, \omega) = \frac{N_c}{4\pi^2\tau^2}e^{-\omega\tau/2}. \quad (7.5)$$

Inverting the Laplace transform (7.3), we find the spectral densities (the integration contour should be to the right of the singularity at $\tau = 0$):

$$\rho_{\pm}(\varepsilon, \omega) = \int_{-i\infty}^{+i\infty} \Pi_{\pm}(\tau, \omega)e^{\varepsilon\tau}\frac{d\tau}{2\pi i}. \quad (7.6)$$

We obtain

$$\rho_+^{(1)}(\varepsilon, \omega) = \frac{N_c}{8\pi^2} \omega \theta\left(\varepsilon - \frac{\omega}{2}\right), \quad \rho_-^{(1)}(\varepsilon, \omega) = \frac{N_c}{4\pi^2} \left(\varepsilon - \frac{\omega}{2}\right) \theta\left(\varepsilon - \frac{\omega}{2}\right). \quad (7.7)$$

The perturbative spectral density $\rho_{\pm}^{(1)}(\varepsilon, \omega)$ describes the contribution of the on-shell quark-antiquark intermediate state with energy ε into the correlator (7.3). The on-shell heavy antiquark has zero energy; the maximum value of p_+ of the on-shell light quark with energy ε is 2ε . This explains the θ -functions in (7.7).

The quark condensate contribution is also important here. It contains the vacuum average

$$\langle \bar{q}(-x)[-x, 0][0, z]q(z) \rangle.$$

This non-collinear quark condensate can be expanded in terms of bilocal condensates [18], the leading term in this expansion is the bilocal quark condensate

$$\langle \bar{q}(0)[0, x]q(x) \rangle = \langle \bar{q}q \rangle f_S(x^2) \quad (7.8)$$

with $x \rightarrow x + z$. This leading term produces the same contribution into $\tilde{\Pi}_{\pm}$:

$$\tilde{\Pi}_{\pm}^{(2)}(x^0, t) = -\frac{1}{4} \langle \bar{q}q \rangle f_S((x + z)^2). \quad (7.9)$$

The expansion of the bilocal quark condensate at small x is

$$f_S(x^2) = 1 + \frac{m_0^2}{16} x^2 + \dots \quad (7.10)$$

where

$$\langle \bar{q}G_{\mu\nu}\sigma^{\mu\nu}q \rangle = m_0^2 \langle \bar{q}q \rangle. \quad (7.11)$$

In general, it can be written as

$$f_S(x^2) = \int \tilde{f}_S(\nu) e^{\nu x^2} d\nu, \quad (7.12)$$

where $\tilde{f}_S(\nu)$ has the meaning of the virtuality distribution function of vacuum quarks. Its moments are expressed via vacuum averages of local operators:

$$\int \tilde{f}_S(\nu) d\nu = 1, \quad \int \tilde{f}_S(\nu) \nu d\nu = \frac{m_0^2}{16}, \quad \dots \quad (7.13)$$

In terms of this function, the quark-condensate contribution into the Fourier-transformed correlators is

$$\Pi_{\pm}^{(2)}(\tau, \omega) = -\frac{\langle \bar{q}q \rangle}{8\tau} \tilde{f}_S\left(\frac{\omega}{2\tau}\right) e^{-\omega\tau/2}. \quad (7.14)$$

In other words, the virtuality distribution function appears directly in the sum rules for the B -meson distribution amplitudes! This is similar to the case of non-diagonal sum rules for the pion distribution amplitude [19].

The local operator expansion (7.10) gives

$$\tilde{f}_S(\nu) = \delta(\nu) - \frac{m_0^2}{16} \delta'(\nu) + \dots \quad (7.15)$$

Of course, a smooth function can always be expanded in derivatives of δ -function, but such an expansion does not tell us much about the shape of this function, unless we sum an infinite number of terms. The bilocal quark condensate (7.8) at large $-x^2$ behaves as

$$f_S(x^2) \sim e^{-\bar{\Lambda}\sqrt{-x^2}}, \quad (7.16)$$

because the Wilson line $[0, x]$ can be considered an HQET heavy-quark propagator, and this vacuum average is the correlator of two HQET heavy–light currents, having B -meson as the lowest-energy intermediate state. Of course, the conditions (7.13), (7.16) don't determine the shape of the distribution function $\tilde{f}_S(\nu)$ in a unique way. They are satisfied by the Bakulev–Mikhailov ansatz [19]

$$\tilde{f}_S(\nu) = N \exp\left(-\frac{\bar{\Lambda}^2}{4\nu} - \sigma\nu\right), \quad (7.17)$$

where the parameters σ , N are determined by (7.13).

Now we equate the theoretical result for the correlators to the result obtained from a phenomenological model of the spectral densities. They have the contribution $\sim \delta(\varepsilon - \bar{\Lambda})$ of the ground-state meson and that of the continuum of excited states. As usual, this contribution is modeled by the perturbative spectral densities (7.7) starting from a continuum threshold energy ε_c :

$$\rho_{\pm}(\varepsilon, \omega) = F^2 \varphi_{\pm}(\omega) \delta(\varepsilon - \bar{\Lambda}) + \rho_{\pm}^{(1)}(\varepsilon, \omega) \theta(\varepsilon - \varepsilon_c). \quad (7.18)$$

With this model, the equality of the phenomenological expression for the correlator and the theoretical one,

$$F^2 \varphi_{\pm}(\omega) e^{-\bar{\Lambda}\tau} + \int_{\varepsilon_c}^{\infty} \rho_{\pm}^{(1)}(\varepsilon, \omega) e^{-\varepsilon\tau} d\varepsilon = \int_0^{\infty} \rho_{\pm}^{(1)}(\varepsilon, \omega) e^{-\varepsilon\tau} d\varepsilon + \Pi_{\pm}^{(2)}(\tau, \omega),$$

becomes

$$F^2 \varphi_{\pm}(\omega) e^{-\bar{\Lambda}\tau} = \int_0^{\varepsilon_c} \rho_{\pm}^{(1)}(\varepsilon, \omega) e^{-\varepsilon\tau} d\varepsilon + \Pi_{\pm}^{(2)}(\tau, \omega), \quad (7.19)$$

where the perturbative spectral density is integrated over the “duality interval” of the ground-state meson (from 0 to the continuum threshold ε_c). We obtain the sum rules for the B -meson distribution amplitudes

$$\begin{aligned} F^2 \varphi_+(\omega) e^{-\bar{\Lambda}\tau} &= \frac{N_c}{8\pi^2\tau} \omega e^{-\omega\tau/2} \delta_0\left(\left(\varepsilon_c - \frac{\omega}{2}\right)\tau\right) - \frac{\langle\bar{q}q\rangle}{8\tau} \tilde{f}_S\left(\frac{\omega}{2\tau}\right) e^{-\omega\tau/2}, \\ F^2 \varphi_-(\omega) e^{-\bar{\Lambda}\tau} &= \frac{N_c}{4\pi^2\tau^2} e^{-\omega\tau/2} \delta_1\left(\left(\varepsilon_c - \frac{\omega}{2}\right)\tau\right) - \frac{\langle\bar{q}q\rangle}{8\tau} \tilde{f}_S\left(\frac{\omega}{2\tau}\right) e^{-\omega\tau/2}. \end{aligned} \quad (7.20)$$

where the functions

$$\delta_n(x) = \theta(x) \left(1 - e^{-x} \sum_{m=0}^n \frac{x^m}{m!} \right) \quad (7.21)$$

describe the effect of subtracting continuum from the perturbative contribution when the spectral density is $\sim \varepsilon^n$. The perturbative contributions vanish at $\omega > 2\varepsilon_c$, because of the properties of the spectral densities (7.7).

Setting $t = 0$ in the correlator (7.1), or integrating (7.2) in $d\omega$, we obtain the well-known sum rule for F^2 :

$$F^2 e^{-\bar{\Lambda}\tau} = \frac{N_c}{2\pi^2\tau^3} \delta_2(\varepsilon_c\tau) - \frac{1}{4} \langle \bar{q}q \rangle f_S(-\tau^2). \quad (7.22)$$

The natural energy scale in this sum rule is

$$k = \left(-\frac{\pi^2}{2N_c} \langle \bar{q}q \rangle \right)^{1/3} \approx 260 \text{ MeV}. \quad (7.23)$$

With $m_0/(4k) = 0.85$, one finds the optimal value of the continuum threshold $\varepsilon_c/k = 3$; then a wide plato at $1/(k\tau) \in [1.7, 2.5]$ exists, and yields $\bar{\Lambda}/k = 1.65$. We divide the sum rules (7.20) by (7.22) and take $1/(k\tau) = 2$ (in the middle of the plato). The results are shown in Fig. 15, 16, where ω is measured in units of k . They are automatically normalized (4.14). Of course, the leading-order sum rules cannot tell us at what scale these distribution amplitudes are normalized; this scale must be low, $\mu \sim 1/\tau$. The perturbative contributions vanish at $\omega > 2\varepsilon_c$. The quark-condensate contribution is the same for both $\varphi_+(\omega)$ and $\varphi_-(\omega)$. Here we used the ansatz (7.17). This contribution gives a sharp peak at low ω^3 . Details of its shape are unknown, but it cannot be wider because of the restriction on the first moment (7.13). This contribution falls off quickly at larger ω . Therefore, the distribution amplitudes at this low μ are only non-zero at $\omega \sim \bar{\Lambda}$, in accord with the expectations of the quark model.

If we neglect the quark-condensate contribution (though this is not a good idea) and also the effect of continuum subtraction⁴, then the perturbative contributions (7.5) suggest the following simple model of distribution amplitudes:

$$\varphi_+(\omega) = \frac{\omega}{\omega_0^2} e^{-\omega/\omega_0}, \quad \varphi_-(\omega) = \frac{1}{\omega_0} e^{-\omega/\omega_0}. \quad (7.24)$$

They are normalized (4.14); from the first moments (4.26) we obtain

$$\omega_0 = \frac{2}{3} \bar{\Lambda}. \quad (7.25)$$

These functions (Fig. 17) have something in common with the Wandzura–Wilczek ones (Fig. 6), but in contrast to them they fall off smoothly at large ω .

³For the pion distribution amplitude, the quark-condensate contribution gives enhancements near $x = 0$ and 1.

⁴If the same procedure is applied to the diagonal sum rule for the pion distribution amplitude, it yields the asymptotic shape.

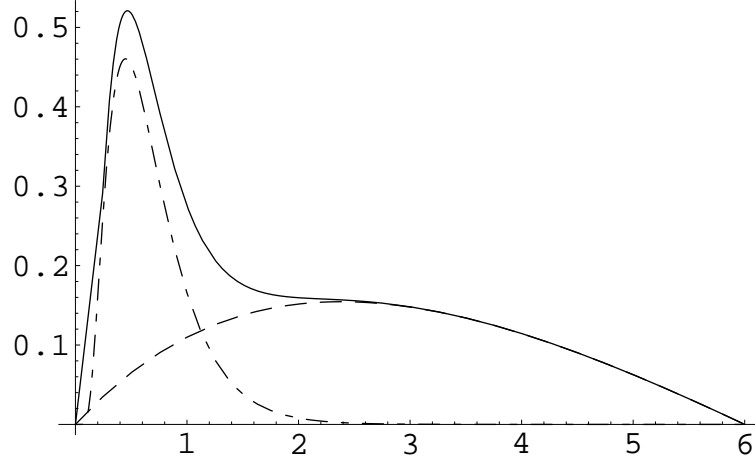


Figure 15: Distribution amplitude $\varphi_+(\omega)$ (solid line), perturbative contribution (dashed line), and quark-condensate contribution (dashed-dotted line)

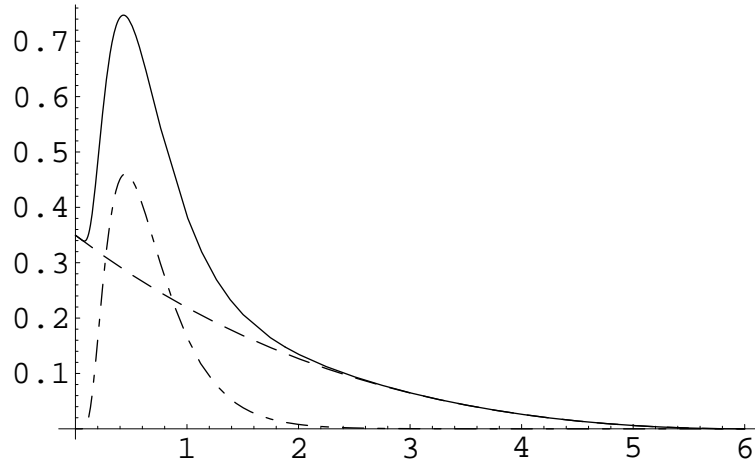


Figure 16: Distribution amplitude $\varphi_-(\omega)$ (solid line), perturbative contribution (dashed line), and quark-condensate contribution (dashed-dotted line)

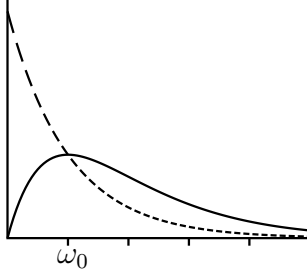


Figure 17: Model distribution amplitudes: $\varphi_+(\omega)$ (solid line) and $\varphi_-(\omega)$ (dashed line)

Radiative corrections to the perturbative spectral density and the dimension-3 quark-condensate contribution for φ_+ were calculated in [9]. With these corrections, one can check that the correlator (7.2) satisfies the evolution equation (6.8). The resulting sum rules should be used together with the sum rules for F^2 with radiative corrections [20]. The perturbative spectral density is

$$\rho_+^{(1)}(\varepsilon, \omega) = \frac{N_c}{8\pi^2} \omega \times \begin{cases} x > 1: & 1 + C_F \frac{\alpha_s}{4\pi} \left[-2 \log^2 \frac{\omega}{\mu} - 4(\log(x-1) + 1) \log \frac{\omega}{\mu} \right. \\ & \quad \left. + 2 \text{Li}_2 \left(\frac{1}{1-x} \right) - \log^2(x-1) \right. \\ & \quad \left. - (2x+3) \log(x-1) + 2x \log x + \frac{7}{12} \pi^2 + 7 \right] \\ x < 1: & 2C_F \frac{\alpha_s}{4\pi} \left[2(\log(1-x) + x) \log \frac{\omega}{\mu} + 2 \log^2(1-x) \right. \\ & \quad \left. + (2x-1) \log(1-x) - x \right] \end{cases} \quad (7.26)$$

where $x = 2\varepsilon/\omega$. Now the spectral density does not vanish at $\varepsilon < \omega/2$, it is just suppressed by $\alpha_s/(4\pi)$. Therefore, the perturbative contribution to the sum rule (7.20) for $\varphi_+(\omega)$ does not vanish at $\omega > 2\varepsilon_c$. It produces a radiative tail $\sim 1/\omega$ with the magnitude of order $\alpha_s/(4\pi)$.

The radiative correction to the dimension-3 quark-condensate contribution cannot be used together with a model of bilocal condensate, because radiative corrections to higher-dimensional contributions are not known. It is vital to use some model of the bilocal quark condensate in the sum rules for the distribution amplitude, because the local operator expansion produces contributions $\delta(\omega)$, $\delta'(\omega)$, \dots , which don't tell us much about the shape of the distribution amplitude. Therefore, the full result for this correction cannot be used in the sum rule for $\varphi_+(\omega)$. Fortunately, the authors of [9] demonstrated that a part of this correction is universal, and exponentialized into the Sudakov factor. We may multiply the bilocal quark-condensate contribution by this Sudakov factor.

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